

GAUSS MAP OF A HARMONIC SURFACE

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ABSTRACT. We prove that the distortion function of the Gauss map of a harmonic surface coincides with the distortion function of the surface. Consequently, Gauss map of a harmonic surface is \mathcal{K} quasiconformal if and only if the surface is \mathcal{K} quasiconformal, provided that the Gauss map is regular or what is shown to be the same, provided that the surface is non-planar.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

An orientation preserving smooth mapping $\varphi : \Omega \rightarrow \Omega'$, between two open domains $\Omega, \Omega' \subset \mathbf{C}$ is called \mathcal{K} ($\mathcal{K} \geq 1$) quasiconformal if dilatation d_f of the Beltrami coefficient $\mu(z) := \varphi_{\bar{z}}/\varphi_z$ satisfies the inequality

$$d_\varphi := |\mu(z)| \leq k := \frac{\mathcal{K} - 1}{\mathcal{K} + 1},$$

or what is the same if

$$\|\nabla\varphi\|^2 \leq \frac{1}{2} \left(\mathcal{K} + \frac{1}{\mathcal{K}} \right) J_\varphi,$$

where $\|\nabla\varphi\|$ is the Hilbert-Schmidt norm defined by

$$\|\nabla\varphi\|^2 := |\varphi_z|^2 + |\varphi_{\bar{z}}|^2$$

and

$$J_\varphi = (|\varphi_z|^2 - |\varphi_{\bar{z}}|^2)$$

is the Jacobian of $\nabla\varphi$.

Note that in our context is enough to assume that φ is smooth mapping, however, the classical definition of quasiconformality assumes weaker conditions (cf. [1, pp. 3, 23–24]). Note also that, in this definition we do not assume injectivity.

1.1. Parametric Surfaces. We define an oriented parametric surface \mathcal{M} in \mathbf{R}^3 to be an equivalence class of mappings $Y = (a, b, c) : D \rightarrow \mathbf{R}^3$ of some domain $D \subset \mathbf{C}$ into \mathbf{R}^3 , where the coordinate functions a, b, c are of class at least $\mathcal{C}^1(D)$. Two such mappings $X : D \rightarrow \mathbf{R}^3$ and $\tilde{X} : \tilde{D} \rightarrow \mathbf{R}^3$, referred to as parametrizations of the surface, are said to be equivalent if there is a \mathcal{C}^1 -diffeomorphism $\phi : \tilde{D} \xrightarrow{\text{onto}} D$ of positive Jacobian determinant such that $\tilde{Y} = Y \circ \phi$. Let us call such ϕ a *change of variables, or reparametrization* of the surface. Furthermore, we assume that the branch (critical) points of \mathcal{M} are isolated. These are the points $(x, y) \in D$ at which the tangent vectors $Y_x = \frac{\partial Y}{\partial x}$, $Y_y = \frac{\partial Y}{\partial y}$ are linearly dependent or equivalently $Y_x \times Y_y \neq 0$ where by \times is denoted the standard vectorial product in the space \mathbf{R}^3 . Equivalently, at the critical points the *Jacobian matrix* $\nabla Y(z)$ has rank at most 1.

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It has full rank 2 at the *regular points*. A surface with no critical points is called an *immersion* or a *regular surface*.

For regular points of the surface we define the distortion function by

$$\mathfrak{D}_Y = \frac{\|Y_x\|^2 + \|Y_y\|^2}{2\|Y_x \times Y_y\|}.$$

A mapping Y is called \mathcal{K} quasiconformal if

$$(1.1) \quad \|Y_x\|^2 + \|Y_y\|^2 \leq \left(\mathcal{K} + \frac{1}{\mathcal{K}}\right) \|Y_x \times Y_y\|, \quad z = x + iy \in D.$$

If $\mathcal{K} = 1$ then (1.1) is equivalent to the system of the equations

$$(1.2) \quad \|Y_x\|^2 = \|Y_y\|^2 \quad \text{and} \quad \langle Y_x, Y_y \rangle = 0,$$

which represent isothermal (conformal) coordinates of the surface M . If u is harmonic and satisfies the system (1.2) then M is a minimal surface.

If Y is an immersion of M and $X = (u, v, w)$ are isothermal coordinates of a smooth surface M , and if $\varphi = X^{-1} \circ Y$, then we have

$$(1.3) \quad \mathfrak{D}_Y = \mathfrak{D}_\varphi = \frac{|\varphi_z|^2 + |\varphi_{\bar{z}}|^2}{|\varphi_z|^2 - |\varphi_{\bar{z}}|^2}.$$

1.2. Gauss map of a surface. Let $X : \Omega \rightarrow \mathbf{R}^3$ be a smooth regular surface and let $M = X(\Omega)$. Let S^2 be the unit sphere in \mathbf{R}^3 and let $\mathbf{n} : \Omega \rightarrow S^2$ be the orientation preserving Gauss map of M defined as follows

$$\mathbf{n}(z) = \frac{X_u \times X_v}{\|X_u \times X_v\|}.$$

If we denote by $f : S^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbf{R}^2$,

$$f(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right)$$

the stereographic projection, then the orientation preserving map $\mathbf{g} := f \circ \mathbf{n}$ is said to be the complex Gauss map of X .

Since f is a conformal mapping, then \mathbf{g} is quasiconformal if and only if \mathbf{n} is quasiconformal with the same constant of quasiregularity. Moreover

$$(1.4) \quad \mathfrak{D}_{\mathbf{n}} = \mathfrak{D}_{\mathbf{g}} = \frac{|\mathbf{g}_z|^2 + |\mathbf{g}_{\bar{z}}|^2}{|\mathbf{g}_z|^2 - |\mathbf{g}_{\bar{z}}|^2}.$$

A smooth enough surface can be parameterized using a isothermal parameterization. Such a parameterization is minimal if the coordinate functions x_k are harmonic, i.e., $\phi_k(\zeta)$ are analytic. A minimal surface can therefore be defined by a triple of analytic functions such that

$$(1.5) \quad \phi_1^2 + \phi_2^2 + \phi_3^2 = 0.$$

The real parameterization is then obtained as

$$(1.6) \quad x_k = \Re \int \phi_k(\zeta) d\zeta.$$

But, for an analytic function p and a meromorphic function q , the triple of functions

$$(1.7) \quad \phi_1(\zeta) = p(1 - q^2)$$

$$(1.8) \quad \phi_2(\zeta) = ip(1 + q^2)$$

$$(1.9) \quad \phi_3(\zeta) = 2pq$$

are analytic as long as p has a zero of order $\geq m$ at every pole of q of order m . This gives a minimal surface in terms of the Enneper-Weierstrass parameterization

$$(1.10) \quad \Re \int \{p(1 - q^2); ip(1 + q^2); 2pq\} d\zeta.$$

It is well known that if the surface is minimal endowed with Enneper-Weierstrass parameterization, then its complex Gauss map is a meromorphic function $\mathbf{g}(w) = -i/q'(w)$. See for example [5, Sec. 9.3] for the above facts.

In this paper, we extend this result to quasiconformal harmonic surfaces by proving the following theorem.

Theorem 1.1. *The dilatation and distortion function of the Gauss map \mathbf{n} of a harmonic surface X coincides with the dilatation and distortion function of the surface itself, provided that the Gauss map is regular. In other words, if*

$$(1.11) \quad \frac{\partial \mathbf{n}}{\partial x}(z) \times \frac{\partial \mathbf{n}}{\partial y}(z) \neq 0,$$

then

$$(1.12) \quad \mathfrak{D}_{\mathbf{n}}(z) = \mathfrak{D}_{\mathbf{X}}(z)$$

and

$$(1.13) \quad d_X(z) = d_{\mathbf{n}}(z).$$

Corollary 1.2. *If a harmonic parametric surface X is \mathcal{K} quasiconformal then its Gauss map is \mathcal{K} quasiconformal, provided that the Gauss map is regular.*

We also prove the following theorem

Theorem 1.3. *The Gauss map of a harmonic surface is regular if and only if the surface is not planar. If the Gauss map of a surface is not regular, then it must be a constant vector.*

Remark 1.4. From Theorem 1.3 we find out that, in Theorem 1.1, instead of the condition "the Gauss map is regular" we can simply say "the surface is non-planar". When we say that the Gauss map is regular we mean that, the cross product $\frac{\partial}{\partial x} \mathbf{n} \times \frac{\partial}{\partial y} \mathbf{n}$ is not identically zero at some open subset of the domain. However, as it can be shown by the example

$$X = \left\{ x, -\frac{x^3}{3} + x \left(\frac{1}{2} + y \right)^2, 1 - x^2 + y + y^2 \right\},$$

that

$$\frac{\partial}{\partial x} \mathbf{n} \times \frac{\partial}{\partial y} \mathbf{n} = 0,$$

for $y = -1/2$. Thus, the branch points of the Gauss normal of harmonic surfaces are not isolated, as in the case of minimal surfaces.

Remark 1.5. The class of minimal surfaces with Enneper-Weierstrass parameterization is a special case of the class of quasiconformal harmonic surfaces. Namely in this case $\mathcal{K} = 1$, because the coordinates are isothermal (conformal). In this case the condition (1.11) reduces to the condition that $\mathbf{g}(w) = -i/q'(w) = \text{const}$, i.e. the Gauss normal is a constant. This implies that the minimal surface is planar.

The class of quasiconformal harmonic mapping between complex domains and two-dimensional surfaces has been very active research of investigation in recent years. For some regularity results of this class we refer to the following papers [9], [8], [10], [14]. For some regularity results of the class of minimal surfaces we refer to the papers of J. C. C. Nitsche [11] and [12].

Recall that by a definition of A. Alarcon and F. J. Lopez [2] a harmonic immersion $X : M \rightarrow \mathbf{R}^3$ is said to be quasiconformal (QC for short) if its orientation preserving Gauss map $\mathbf{n} : M \rightarrow S^2$ is quasiconformal (or equivalently, if \mathbf{g} is quasiconformal). Among other special features, quasiconformal harmonic immersions are quasiminimal in the sense of Osserman [13]. In this case, X is said to be a harmonic QC parameterization of the harmonic surface $X(M)$. Notice also this, a harmonic surface has no elliptic points by the maximum principle for harmonic functions. In other words, its Gauss curvature is nowhere positive. Let w a harmonic diffeomorphism of the unit disk onto itself which is not quasiconformal. Let $X(z) = (\Re(w), \Im(w), 0)$. Then $\mathbf{n} = (0, 0, 1)$ and therefore it is a 1-quasiconformal mapping. This mean that the fact that the condition "the Gauss map is regular" is important in Theorem 1.1. In other words, two above definitions of quasiconformality are equivalent, provided that the Gauss map is regular (up to branch points) or what is the same, if the surface is not planar.

2. PROOFS

Proof of Theorem 1.1. Let

$$X(x, y) = (a(x, y), b(x, y), c(x, y)).$$

Without loss of generality we can assume that $a(z) = x$. Namely, let ϕ be an analytic mapping of the unit disk into \mathbf{C} such that

$$\phi(z) = a(z) + i\tilde{a}(z).$$

Here $\tilde{a}(z)$ is the harmonic conjugate of $a(z)$. Take instead of

$$X(z) = (a(z), b(z), c(z))$$

$$\tilde{X} = X \circ \phi^{-1}(z)$$

in some neighborhood D of some nonsingular point z of $\phi'(z)$. Then the first coordinate of \tilde{f} is x in D .

Let

$$\mathbf{n}(z) = \frac{X_x \times X_y}{\|X_x \times X_y\|}.$$

Then $\mathbf{n}(z)$ is given by

$$\left\{ \frac{c_y b_x - b_y c_x}{\sqrt{b_y^2 + c_y^2 + (c_y b_x - b_y c_x)^2}}, -\frac{c_y}{\sqrt{b_y^2 + c_y^2 + (c_y b_x - b_y c_x)^2}}, \frac{b_y}{\sqrt{b_y^2 + c_y^2 + (c_y b_x - b_y c_x)^2}} \right\}.$$

Define

$$P = \frac{d}{dx} \mathbf{n}(z), \quad Q = \frac{d}{dy} \mathbf{n}(z).$$

The distortion function is

$$\mathfrak{D}_{\mathbf{n}} = \frac{|P|^2 + |Q|^2}{|P \times Q|}.$$

Then

$$(2.1) \quad |P|^2 + |Q|^2 = \frac{N}{G^4}$$

and

$$(2.2) \quad |P \times Q| = \frac{M}{G^3}$$

where

$$M = (c_y^2 (b_{xy}^2 - b_{yy}b_{xx}) + b_y c_y (-2b_{xy}c_{xy} + c_{yy}b_{xx} + b_{yy}c_{xx}) + b_y^2 (c_{xy}^2 - c_{yy}c_{xx})),$$

$$G = \sqrt{b_y^2 + c_y^2 + (c_y b_x - b_y c_x)^2}$$

and

$$\begin{aligned} N = & c_y^4 (b_{xy}^2 + b_{xx}^2) \\ & + b_y^2 (c_{yy}^2 (1 + b_x^2 + c_x^2) - 2b_y c_{yy} b_x c_{xy} + (1 + b_y^2 + b_x^2 + c_x^2) c_{xy}^2 - 2b_y b_x c_{xy} c_{xx} + b_y^2 c_{xx}^2) \\ & - 2c_y^3 (b_{yy} c_x b_{xy} + c_x b_{xy} b_{xx} + b_y (b_{xy} c_{xy} + b_{xx} c_{xx})) \\ & - 2b_y c_y \left[(1 + b_x^2 + c_x^2) b_{xy} c_{xy} + b_{yy} (c_{yy} (1 + b_x^2 + c_x^2) - b_y b_x c_{xy}) \right. \\ & \left. + b_y^2 (b_{xy} c_{xy} + b_{xx} c_{xx}) - b_y (c_{yy} (b_x b_{xy} - c_x c_{xy}) - c_x c_{xy} c_{xx} + b_x (c_{xy} b_{xx} + b_{xy} c_{xx})) \right] \\ & + c_y^2 \left[b_{yy}^2 (1 + b_x^2 + c_x^2) + (1 + b_x^2 + c_x^2) b_{xy}^2 + 2b_y b_{yy} (-b_x b_{xy} + c_x c_{xy}) \right. \\ & \left. + b_y^2 (b_{xy}^2 + c_{xy}^2 + b_{xx}^2 + c_{xx}^2) + 2b_y (c_{yy} c_x b_{xy} - b_x b_{xy} b_{xx} + c_x (c_{xy} b_{xx} + b_{xy} c_{xx})) \right]. \end{aligned}$$

Let

$$\begin{aligned} A &= b_{xy}^2 - b_{yy}b_{xx}, \quad B = b_{xy}c_{xy} + b_{xx}c_{xx}, \quad C = c_{xy}^2 - c_{yy}c_{xx} \\ \delta &= (1 + b_x^2 + c_x^2). \end{aligned}$$

Then, since $\Delta b = \Delta c = 0$ we obtain

$$M = (c_y^2 A - 2b_y c_y B + b_y^2 C)$$

and

$$\begin{aligned} N = & c_y^4 A \\ & + b_y^2 (c_{yy}^2 \delta - 2b_y c_{yy} b_x c_{xy} + (1 + b_y^2 + b_x^2 + c_x^2) c_{xy}^2 - 2b_y b_x c_{xy} c_{xx} + b_y^2 c_{xx}^2) \\ & - 2c_y^3 (b_{yy} c_x b_{xy} + c_x b_{xy} b_{xx} + b_y B) \\ & - 2b_y c_y \left[(\delta + b_y^2) B - b_y b_x b_{yy} c_{xy} - b_y (c_{yy} (b_x b_{xy} - c_x c_{xy}) - c_x c_{xy} c_{xx} + b_x (c_{xy} b_{xx} + b_{xy} c_{xx})) \right] \\ & + c_y^2 \left[\delta A + 2b_y b_{yy} (-b_x b_{xy} + c_x c_{xy}) \right. \\ & \left. + b_y^2 (A + C) + 2b_y (c_{yy} c_x b_{xy} - b_x b_{xy} b_{xx} + c_x (c_{xy} b_{xx} + b_{xy} c_{xx})) \right]. \end{aligned}$$

Further, since $\Delta b = 0$

$$\begin{aligned} N &= c_y^4 A \\ &\quad + b_y^2 (c_{yy}^2 \delta - 2b_y c_{yy} b_x c_{xy} + (1 + b_y^2 + b_x^2 + c_x^2) c_{xy}^2 - 2b_y b_x c_{xy} c_{xx} + b_y^2 c_{xx}^2) - 2c_y^3 b_y B \\ &\quad - 2b_y c_y \left[(\delta + b_y^2) B - b_y b_x b_{yy} c_{xy} - b_y (c_{yy} (b_x b_{xy} - c_x c_{xy}) - c_x c_{xy} c_{xx} + b_x (c_{xy} b_{xx} + b_{xy} c_{xx})) \right] \\ &\quad + c_y^2 \left[\delta A + 2b_y b_{yy} (c_x c_{xy}) + b_y^2 (A + C) + 2b_y (c_{yy} c_x b_{xy} + c_x (c_{xy} b_{xx} + b_{xy} c_{xx})) \right] \end{aligned}$$

and

$$\begin{aligned} N &= c_y^4 A + b_y^2 (c_{yy}^2 \delta + (1 + b_y^2 + b_x^2 + c_x^2) c_{xy}^2 + b_y^2 c_{xx}^2) - 2c_y^3 b_y B \\ &\quad - 2b_y c_y \left[(\delta + b_y^2) B - b_y b_x b_{yy} c_{xy} - b_y (c_{yy} (b_x b_{xy} - c_x c_{xy}) - c_x c_{xy} c_{xx} + b_x (c_{xy} b_{xx} + b_{xy} c_{xx})) \right] \\ &\quad + c_y^2 \left[\delta A + 2b_y b_{yy} (c_x c_{xy}) + b_y^2 (A + C) + 2b_y (c_{yy} c_x b_{xy} + c_x (c_{xy} b_{xx} + b_{xy} c_{xx})) \right]. \end{aligned}$$

Finally

$$\begin{aligned} N &= c_y^4 A + b_y^2 (c_{yy}^2 \delta + (1 + b_y^2 + b_x^2 + c_x^2) c_{xy}^2 + b_y^2 c_{xx}^2) - 2c_y^3 b_y B \\ &\quad - 2b_y c_y \left[(\delta + b_y^2) B \right] + c_y^2 \left[\delta A + b_y^2 (A + C) \right] \\ &= c_y^4 A + b_y^2 ((b_y^2 + \delta) C) - 2c_y^3 b_y B - 2b_y c_y \left[(\delta + b_y^2) B \right] + c_y^2 \left[\delta A + b_y^2 (A + C) \right] \\ &= c_y^2 (c_y^2 + \delta + b_y^2) A + b_y^2 (b_y^2 + \delta + c_y^2) C - 2b_y c_y (\delta + b_y^2 + c_y^2) C \\ &= \gamma (c_y^2 A + b_y^2 C - 2b_y c_y C) \end{aligned}$$

where

$$\gamma = 1 + b_x^2 + b_y^2 + c_x^2 + c_y^2.$$

As

$$\mathfrak{D}_{\mathbf{n}} = \frac{N}{GM}$$

we obtain

$$\begin{aligned} \mathfrak{D}_{\mathbf{n}} &= \frac{\gamma (c_y^2 A + b_y^2 C - 2b_y c_y C)}{G (c_y^2 A + b_y^2 C - 2b_y c_y C)} \\ &= \frac{1 + b_y^2 + c_y^2 + b_x^2 + c_x^2}{\sqrt{b_y^2 + c_y^2 + (c_y b_x - b_y c_x)^2}} = \mathfrak{D}_{\mathbf{x}}. \end{aligned}$$

The last part of the theorem follows from the formulas (1.3) and (1.4). \square

Corollary 2.1 (Berenstein theorem for nonparametric harmonic surfaces). *If the harmonic nonparametric surface over \mathbf{R}^2 is quasiconformal, then the surface is planar.*

Proof. From the previous theorem we find out that the Gauss map is quasiconformal. Then by a theorem of L. Simon [15], X must be planar. \square

Corollary 2.2. *If $X = (a, b, c)$ is a harmonic surface then*

$$\mathfrak{D}_{\mathbf{n}} = \frac{|\nabla a|^2 + |\nabla b|^2 + |\nabla c|^2}{2\sqrt{(b_y a_x - a_y b_x)^2 + (-c_y a_x + a_y c_x)^2 + (c_y b_x - b_y c_x)^2}}.$$

Proof of Theorem 1.3. As in the proof of Theorem 1.1, we can assume that the first coordinate of the surface is x . Further, by using the same notation

$$A = b_{xy}^2 - b_{yy}b_{xx}, \quad B = b_{xy}c_{xy} + b_{xx}c_{xx}, \quad C = c_{xy}^2 - c_{yy}c_{xx}$$

$$G = \sqrt{b_y^2 + c_y^2 + (c_y b_x - b_y c_x)^2},$$

and

$$M = (c_y^2 A - 2b_y c_y B + b_y^2 C),$$

we obtain that

$$\frac{\partial}{\partial x} \mathbf{n} \times \frac{\partial}{\partial y} \mathbf{n} = 0$$

if and only if $M = 0$. The last is equivalent with

$$(|c_y|\sqrt{A} - |b_y|\sqrt{C})^2 + 2(|b_y||c_y|\sqrt{AC} - b_y c_y C) = 0.$$

It is an elementary application of Cauchy-Schwarz inequality that

$$\sqrt{AC} \geq B.$$

Moreover

$$AC = B^2$$

if and only if

$$b_{xx}c_{xx} = b_{xy}c_{xy}.$$

Thus $M = 0$ if and only if

$$(|c_y|\sqrt{A} - |b_y|\sqrt{C}) = 0$$

and

$$|b_y||c_y|\sqrt{AC} - b_y c_y C = 0.$$

Thus

$$b_y \geq 0, \quad c_y \geq 0,$$

$$b_{xx}c_{xx} = b_{xy}c_{xy}$$

and

$$c_y\sqrt{A} - b_y\sqrt{C} = 0.$$

The cases $b_y = 0$ or $c_y = 0$ are trivial. So assume that $b_y \neq 0$ and $c_y \neq 0$. Combining the last two equalities we arrive at equality

$$c_y b_{yy} = b_y c_{yy}.$$

It follows that

$$\frac{d}{dy} \frac{b_y}{c_y} = 0$$

i.e.

$$\frac{b_y}{c_y} = \lambda(x),$$

for some real function $\lambda(x)$ depending only on x . Since b_y and c_y are real analytic, it follows that λ is a real analytic function, i.e.

$$\lambda(x) = \sum_{n=0}^{\infty} \lambda_n x^n.$$

Further

$$c_y = \sum_{n=0}^{\infty} (a_n z^n + b_n \bar{z}^n)$$

and therefore

$$b_y(z) = \sum_{n=0}^n \lambda_n x^n \sum_{n=0}^{\infty} (a_n z^n + b_n \bar{z}^n).$$

Let us show that $\lambda_n = 0$ for $n \geq 2$.

We will use the following well-known fact. Any harmonic function has (locally) a unique representation as a sum of homogeneous harmonic polynomials $\alpha_n z^n + \beta_n \bar{z}^n$. Since $\lambda_1 x(a_1 z + b_1 \bar{z})$ is the only possible harmonic polinom of degree 2 in the expression for b_y , it must be $a_1 z + b_1 \bar{z} = 2a_1 y$. Further

$$\sum_{k=0}^{n-1} \lambda_{n-k} x^{n-k} (a_k z^k + b_k \bar{z}^k)$$

is a harmonic polinom of degree n and is therefore equal to

$$\alpha_n z^n + \beta_n \bar{z}^n.$$

Thus

$$\sum_{k=0}^{n-1} \lambda_{n-k} (z + \bar{z})^{n-k} (a'_k z^k + b'_k \bar{z}^k) = \alpha_n z^n + \beta_n \bar{z}^n.$$

Since the representation

$$\sum_{i,j} q_{ij} z^i \bar{z}^j$$

is unique, it follows that for $n \geq 2$, $\lambda_n = 0$. Hence

$$\frac{b_y}{c_y} = \lambda_1 x + \lambda_0.$$

By a similar argument we find that, if $\lambda_1 \neq 0$, then $c_y = \omega y + \nu$. In this case

$$c(x, y) = \frac{\omega}{2}(y^2 - x^2) + \nu y + \nu_1 x + \nu_0$$

and

$$b_y = (\lambda_1 x + \lambda_0)(\omega y + \nu).$$

Thus

$$b(x, y) = \frac{\omega}{2}(\lambda_1 x + \lambda_0)(y + \nu/\omega)^2 - \frac{\omega}{6\lambda_1^2}(\lambda_1 x + \lambda_0)^3.$$

However b and c obtained in this case do not satisfy the equation $M \equiv 0$ in an open set. Namely

$$M = \lambda_1^4(\nu + \omega y)^8 \neq 0.$$

It remains to consider the case $\lambda_1 = 0$. Then

$$b(z) = \lambda_0 c(z) + \nu(x),$$

implying that $\nu(x) = \nu_0 + \nu_1 x$. Thus

$$X(x) = (x, \lambda_0 c(z) + \nu_0 + \nu_1 x, c(z)).$$

Thus the Gauss map of the surface X is

$$\left\{ \frac{\nu_1}{\sqrt{2 + \nu_1^2}}, -\frac{1}{\sqrt{2 + \nu_1^2}}, \frac{1}{\sqrt{2 + \nu_1^2}} \right\}$$

implying that the surface is planar. □

2.1. An open problem. Whether Berenstien theorem is true for quasiconformal parametric harmonic surfaces?

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